GLOBAL EXPONENTIAL PERIODICITY FOR DISCRETE-TIME HOPFIELD NEURAL NETWORKS WITH FINITE DISTRIBUTED DELAYS AND IMPULSES

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The discrete counterpart of a class of Hopfield neural networks with periodic integral impulsive conditions and finite distributed delays is introduced. The continuation theorem of coincidence degree theory is used to obtain a sufficient condition for the existence of a periodic solution of the discrete system considered. By introducing an appropriate Lyapunov functional a sufficient condition is obtained for the uniqueness and global exponential stability of the periodic solution.

Keywords: Discrete Hopfield neural networks; Finite distributed delays; Periodic impulses; Global exponential periodicity.

1. Introduction

In the present paper we introduce the discrete counterpart of a class of Hopfield neural networks with periodic integral impulsive conditions and finite distributed delays. We apply the continuation theorem of coincidence degree theory (Gaines and Mawhin [7]) to obtain a sufficient condition for the existence of a periodic solution of the discrete system considered. By introducing an appropriate Lyapunov functional we derive a sufficient condition for the uniqueness and global exponential stability of the periodic solution. For works proving the existence of a periodic solution of differential and difference equations by the coincidence degree theory the reader can see Fan and Agarwal [3], Fan and Wang [4,5], Fan *et al.* [6], Li [9], Li and Kuang [10], Li and Lu [11]. In particular, in Li and Lu [11] the existence of a periodic solution of Hopfield-type neural network with impulses is proved. In Zhou *et al.* [14] one proves the existence of a periodic solution of a discrete-time analogue of a bidirectional associative memory (BAM) neural network with periodic coefficients and finite distributed delays without impulses.

2. Statement of the problem. Main results

We consider a class of Hopfield neural networks with periodic integral impulsive conditions and finite distributed delays, which are formulated in the form of a system of impulsive delay differential equations

$$\frac{dx_i}{dt} = -a_i(t)x_i(t) + \sum_{j=1}^{m} b_{ij}(t) f_j\left(\int_0^{\omega} g_{ij}(s)x_j(t-s)\,ds\right) + I_i(t), \quad t \neq t_k, \tag{1}$$

$$\begin{aligned} \Delta x_i(t_k) &\equiv x_i(t_k+0) - x_i(t_k) \\ &= -\gamma_{ik}x_i(t_k) + \sum_{j=1}^{m} B_{ijk}\Phi_j\left(\int_0^{\omega} c_{ij}(s)x_j(t_k-s)\,ds\right) + \alpha_{ik}, \\ & i=\overline{1,m}, \quad k\in\mathbb{Z}, \end{aligned} \tag{2}$$

where m is the number of neurons in the network, $x_i(t)$ is the state of the i-th neuron at time t, $a_i(t) > 0$ is the rate at which the i-th neuron resets its state when isolated from the system, $b_{ij}(t)$ is the synaptic connection weight from the j-th neuron to the i-th one, $f_j(\cdot)$ are signal transmission functions of the j-th neuron, ω is the maximum transmission delay from one neuron to another, $g_{ij}(\cdot)$ and $c_{ij}(\cdot)$ are nonnegative delay kernels, $I_i(t)$ is the external input to the i-th neuron, t_k ($k \in \mathbb{Z}$) are the instants of impulse effect which form a strictly increasing sequence, γ_{ik} $(i = \overline{1,m},\ k \in \mathbb{Z})$ are positive constants.

We assume that the above system (1), (2) satisfies the following periodicity conditions: $a_i(t)$, $b_{ij}(t)$, $I_i(t)$ are ω-periodic in t; $t_{k+p} = t_k + \omega$, $\gamma_{i,k+p} = \gamma_{ik}$, $B_{ij,k+p} = B_{ijk}$, $\alpha_{i,k+p} = \alpha_{ik}$. Without loss of generality we can assume that

$$0 < t_1 < t_2 < \cdots < t_p < \omega.$$

The Hopfield neural network (1) is similar to the bidirectional associative memory neural network considered in Zhou *et al.* [14]

Combining some ideas of Mohamad and Gopalsamy [13], Akça *et al.* [1], Zhou *et al.* [14] we shall formulate the discrete counterpart of system (1), (2). For a positive integer N we choose the discretization step $h = \omega/N$. For the moment we assume N so large that

$$h < \min_{k=\overline{1,p}} (t_{k+1} - t_k).$$

Then each interval $[nh, (n+1)h]$ contains at most one instant of impulse effect t_k.

For convenience we denote $n = [t/h]$, the greatest integer in t/h, and $n_k = [t_k/h]$. Clearly, we will have $n_{k+p} = n_k + N$ for all $k \in \mathbb{Z}$.

Let $n \in \mathbb{Z}$, $n \neq n_k$. This means that the interval $[nh, (n+1)h]$ contains no instant of impulse effect t_k.

We approximate the integral term in (1) by a sum:

$$\int_0^\omega g_{ij}(s) x_j(t-s)\, ds \approx \sum_{\ell=1}^{N} g_{ij}(\ell h) x_j\left((n-\ell)h\right) \varphi(h),$$

where $\varphi(h) = h + O(h^2)$.

Next we approximate the differential equation (1) on the interval $[nh, (n+1)h]$ by

$$\frac{dx_i}{dt} + a_i(nh)x_i(t) = I_i(nh) + \sum_{j=1}^{m} b_{ij}(nh) f_j \left(\sum_{\ell=1}^{N} g_{ij}(\ell h) x_j\left((n-\ell)h\right)\varphi(h) \right).$$

We multiply both sides of this equation by $\exp\left(a_i(nh)t\right)$ and integrate over the interval $[nh, (n+1)h]$. Thus we obtain

$$\begin{aligned} x_i\left((n+1)h\right) - x_i(nh) &= -\left(1 - e^{-a_i(nh)h}\right) x_i(nh) \\ &+ \frac{1 - e^{-a_i(nh)h}}{a_i(nh)} \left\{ I_i(nh) + \sum_{j=1}^{m} b_{ij}(nh) f_j \left(\sum_{\ell=1}^{N} g_{ij}(\ell h) x_j\left((n-\ell)h\right)\varphi(h) \right) \right\}. \end{aligned} \tag{3}$$

Henceforth by abuse of notation we write $x_i(n) = x_i(nh)$ and define $\Delta x_i(n) = x_i(n+1) - x_i(n)$ $(i = \overline{1,m},\ n \in \mathbb{Z})$. For convenience we adopt the notations:

$$A_i(n) = 1 - e^{-a_i(nh)h} \qquad (i = \overline{1,m},\ n \in \mathbb{Z} \setminus \{n_k\}_{k\in\mathbb{Z}}),$$
$$I_i(n) = \frac{1 - e^{-a_i(nh)h}}{a_i(nh)} I_i(nh) \qquad (i = \overline{1,m},\ n \in \mathbb{Z} \setminus \{n_k\}_{k\in\mathbb{Z}}),$$
$$b_{ij}(n) = \frac{1 - e^{-a_i(nh)h}}{a_i(nh)} b_{ij}(nh) \qquad (i,j = \overline{1,m},\ n \in \mathbb{Z} \setminus \{n_k\}_{k\in\mathbb{Z}}),$$
$$g_{ij}(\ell) = g_{ij}(\ell h)\varphi(h) \qquad (i,j = \overline{1,m},\ \ell = \overline{1,N}).$$

Clearly, we have $0 < A_i(n) < 1$. In particular, if $a_i(t) < \frac{1}{\omega}$, then $A_i(n) < \frac{1}{N}$.

With the above notation equation (3) takes the form

$$\Delta x_i(n) = -A_i(n)x_i(n) + I_i(n) + \sum_{j=1}^{m} b_{ij}(n) f_j \left(\sum_{\ell=1}^{N} g_{ij}(\ell) x_j(n-\ell) \right), \tag{4}$$
$$i = \overline{1,m}, \quad n \neq n_k.$$

Next, for $n = n_k$ the interval $[nh, (n+1)h]$ contains the instant of impulse effect t_k. On this interval we approximate the impulse condition (2) by

$$\Delta x_i(n_k) = -\gamma_{ik} x_i(n_k) + \alpha_{ik} + \sum_{j=1}^{m} B_{ijk} \Phi_j \left(\sum_{\ell=1}^{N} c_{ij}(\ell) x_j(n_k-\ell) \right), \tag{5}$$
$$i = \overline{1,m}, \quad k \in \mathbb{Z},$$

where

$$c_{ij}(\ell) = c_{ij}(\ell h)\varphi(h) \qquad (i,j = \overline{1,m}, \quad \ell = \overline{1,N}).$$

For uniformity of notation we define

$$A_i(n_k) = \gamma_{ik}, \quad I_i(n_k) = \alpha_{ik} \qquad (i = \overline{1,m},\ k \in \mathbb{Z}).$$

Now the difference system (4), (5) can be written in operator form as

$$\Delta x = Hx, \tag{6}$$

where

$$(Hx)_i(n) = -A_i(n)x_i(n) + I_i(n) \tag{7}$$
$$+ \begin{cases} \sum\limits_{j=1}^{m} b_{ij}(n) f_j \left(\sum\limits_{\ell=1}^{N} g_{ij}(\ell) x_j(n-\ell) \right), & n \neq n_k, \\ \sum\limits_{j=1}^{m} B_{ijk} \Phi_j \left(\sum\limits_{\ell=1}^{N} c_{ij}(\ell) x_j(n_k-\ell) \right), & n = n_k. \end{cases}$$

In order to formulate our assumptions, we need some more notation:

$$I_N = \{0,\, 1,\, \dots,\, N-1\},$$

$$\underline{A}_i = \min_{n\in I_N} A_i(n), \quad \overline{A}_i = \sum_{n=0}^{N-1} A_i(n), \qquad i = \overline{1,m}.$$

Now we introduce the following conditions:

H1. $A_i(n+N) = A_i(n)$, $I_i(n+N) = I_i(n)$ for $i = \overline{1,m}$, $n \in \mathbb{Z}$; $n_k \in \mathbb{Z}$ for all $k \in \mathbb{Z}$ and $n_{k+p} = n_k + N$; $b_{ij}(n+N) = b_{ij}(n)$ $(n \neq n_k)$, $B_{ij,k+p} = B_{ijk}$ $(k \in \mathbb{Z})$ for $i,j = \overline{1,m}$.

H2. $\underline{A}_i > 0$, $\overline{A}_i < 1$ for $i = \overline{1,m}$.

H3. The functions $f_j(\cdot)$, $\Phi_j(\cdot)$ $(j = \overline{1,m})$ are bounded on $\mathbb{R}$ and there exist positive constants M_j and L_j such that

$$|f_j(x) - f_j(y)| \leq M_j|x-y|, \quad |\Phi_j(x) - \Phi_j(y)| \leq L_j|x-y|$$

for all $x, y \in \mathbb{R}$.

H4. $g_{ij}(\ell) \geq 0$, $c_{ij}(\ell) \geq 0$ for $i,j = \overline{1,m}$, $\ell = \overline{1,N}$.

We again introduce some notation:

$$\overline{I}_i = \max_{n\in I_N} |I_i(n)|, \qquad i = \overline{1,m},$$

$$\overline{b}_{ij} = \sup_{n\neq n_k} |b_{ij}(n)|, \quad \overline{B}_{ij} = \max_{k=\overline{1,p}} |B_{ijk}|, \qquad i,j = \overline{1,m}.$$

For an N-periodic sequence $v(n)$ we denote $\tilde{v} = \frac{1}{N}\sum\limits_{n=0}^{N-1} v(n)$; for $i = \overline{1,m}$

$$\rho_i = \overline{I}_i + \frac{1}{N}\sum_{j=1}^{m}\left[(N-p)\overline{b}_{ij}|f_j(0)| + p\overline{B}_{ij}|\Phi_j(0)|\right].$$

Next we denote

$$\mathcal{M}_j = \max\{L_j,\, M_j\}, \qquad j = \overline{1,m},$$

$$G_{ij} = \sum_{\ell=1}^{N} g_{ij}(\ell), \quad C_{ij} = \sum_{\ell=1}^{N} c_{ij}(\ell),\;\; i,j = \overline{1,m},$$

$$\mathcal{B}_{ij} = \max\{\overline{b}_{ij},\, \overline{B}_{ij}\}, \quad \mathcal{G}_{ij} = \max\{G_{ij},\, C_{ij}\},\;\; i,j = \overline{1,m}.$$

We introduce the $m \times m$ matrices

$$A = \operatorname{diag}\left(\frac{\underline{A}_i\left(1-\overline{A}_i\right)}{1+N\underline{A}_i},\; i = \overline{1,m}\right), \qquad B = \left(\mathcal{B}_{ij}\mathcal{M}_j\mathcal{G}_{ij}\right)_{i,j=1}^{m}.$$

Then we introduce the conditions

H5. $\min\limits_{i=\overline{1,m}}\left(\tilde{A}_i - \mathcal{M}_i \sum\limits_{j=1}^{m} \mathcal{B}_{ji}\mathcal{G}_{ji}\right) > 0.$

H6. $\underline{A}_i > \mathcal{M}_i \sum\limits_{j=1}^{m} \mathcal{B}_{ji}\mathcal{G}_{ji}$ for $i = \overline{1,m}$.

H7. The matrix $A - B$ is an M-matrix (Fiedler [2], Horn and Johnson [8]).

Theorem 2.1. *Suppose that conditions **H1**–**H5**, **H7** hold. Then the equation* (6) *has at least one N-periodic solution.*

Sketch of the proof. We shall prove Theorem 2.1 using Mawhin's continuation theorem (Gaines and Mawhin [7], p. 40). To state this theorem we need some preliminaries:

Let $\mathbb{X}, \mathbb{Y}$ be real Banach spaces, $L:\ \text{Dom}\ L \subset \mathbb{X} \longrightarrow \mathbb{Y}$ be a linear mapping, and $H:\ \mathbb{X} \longrightarrow \mathbb{Y}$ be a continuous mapping. The mapping L will be called a Fredholm mapping of index zero if $\dim \text{Ker}\ L = \text{codim}\ \text{Im}\ L < +\infty$ and $\text{Im}\ L$ is closed in $\mathbb{Y}$. If L is a Fredholm mapping of index zero and there exist continuous projectors $P:\ \mathbb{X} \longrightarrow \mathbb{X}$ and $Q:\ \mathbb{Y} \longrightarrow \mathbb{Y}$ such that $\text{Im}\ P = \text{Ker}\ L$, $\text{Ker}\ Q = \text{Im}\ L = \text{Im}\,(I-Q)$, then the mapping $L|_{\text{Dom}\ L \cap \text{Ker}\ P} : (I-P)\mathbb{X} \longrightarrow \text{Im}\ L$ is invertible. We denote the inverse of this mapping by K_P. If Ω is an open bounded subset of $\mathbb{X}$, the mapping H will be called L-compact on $\overline{\Omega}$ if $QH(\overline{\Omega})$ is bounded and $K_P(I-Q)H:\ \overline{\Omega} \longrightarrow \mathbb{X}$ is compact. Since $\text{Im}\ Q$ is isomorphic to $\text{Ker}\ L$, there exists an isomorphism $J:\ \text{Im}\ Q \longrightarrow \text{Ker}\ L$.

Now Mawhin's continuation theorem can be stated as follows.

Lemma 2.1. *Let L be a Fredholm mapping of index zero, let $\Omega \subset \mathbb{X}$ be an open bounded set and let $H:\ \mathbb{X} \longrightarrow \mathbb{Y}$ be a continuous operator which is L-compact on $\overline{\Omega}$. Assume that the following conditions hold:*

(*a*) *for each $\lambda \in (0,\,1)$, $x \in \partial\Omega \cap \text{Dom}\, L$, $Lx \neq \lambda Hx$;*

(*b*) *for each $x \in \partial\Omega \cap \text{Ker}\, L$, $QHx \neq 0$;*

(*c*) $\deg\,(JQH,\ \Omega \cap \text{Ker}\, L,\ 0) \neq 0$*, where* $\deg(\cdot)$ *is the Brouwer degree [12]. Then the equation $Lx = Hx$ has at least one solution in $\overline{\Omega} \cap \text{Dom}\, L$.*

Let us choose $\mathbb{X} = \mathbb{Y} = \{x(n) = (x_1(n), x_2(n), \ldots, x_m(n))^T :\ x(n+N) = x(n),\ n \in \mathbb{Z}\}$. If we define $|x_i| = \max\limits_{n \in I_N} |x_i(n)|$, $\|x\| = \sum\limits_{i=1}^{m} |x_i|$, then $\mathbb{X}$ is a Banach space with the norm $\|\cdot\|$. For $x \in \mathbb{X}$, let Hx be defined by (7), $Lx = \Delta x$ and

$$Px = Qx = \tilde{x} = (\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_m)^T.$$

Then Ker $L = \{x \in \mathbb{X} : \ x = h \in \mathbb{R}^m\}$ (vectors with components independent of n), Im $L = \{x \in \mathbb{X} : \sum_{n=0}^{N-1} x_i(n) = 0, \ i = \overline{1,m}\}$ is a closed set in $\mathbb{X}$, and codim $L = m$. Thus L is a Fredholm mapping of index zero. It is easy to see that P and Q are continuous projectors and Im $P =$ Ker L, Im $L =$ Ker $Q =$ Im$\,(I - Q)$, and H is L-compact on $\bar{\Omega}$ for any bounded set $\Omega \subset \mathbb{X}$. Moreover, in condition (c) of Lemma 2.1 the isomorphism J can be taken as the identity operator I.

First for the solutions x of the operator equation $Lx = \lambda Hx$ for $\lambda \in (0,\, 1)$, that is,

$$\Delta x_i(n) = \lambda (Hx)_i(n), \quad n \in I_N, \ i = \overline{1,m},$$

after lengthy calculations we derive the estimate

$$\frac{\underline{A}_i(1 - \overline{A}_i)}{1 + N\underline{A}_i}|x_i| - \sum_{j=1}^{m} \mathcal{B}_{ij}\mathcal{M}_j\mathcal{G}_{ij}|x_j| \le \rho_i. \tag{8}$$

If we introduce the vectors $|\boldsymbol{x}| = (|x_1|, |x_2|, \dots, |x_m|)^T$ and $\boldsymbol{\rho} = (\rho_1, \rho_2, \dots, \rho_m)^T$, then the system of inequalities (8) for $i = \overline{1,m}$ can be written in a matrix form

$$(A - B)|\boldsymbol{x}| \le \boldsymbol{\rho}, \tag{9}$$

where the matrices A and B were introduced in §2. By virtue of condition **H7** the inequality (9) implies

$$|\boldsymbol{x}| \le (A - B)^{-1}\boldsymbol{\rho}.$$

If $(A - B)^{-1}\boldsymbol{\rho} = (C_1^*, C_2^*, \dots, C_m^*)^T$, this means that the components of each solution of $\Delta x = \lambda Hx$ satisfy $|x_i| \le C_i^*$. If we denote $C^* = \sum_{i=1}^{m} C_i^*$, then each solution of $\Delta x = \lambda Hx$ satisfies $\|x\| \le C^*$.

Now we take $\Omega = \{x \in \mathbb{X} : \ \|x\| < C\}$, where $C > C^*$ will be chosen later. Obviously Ω satisfies condition (a) of Lemma 2.1.

Now let $x \in \partial\Omega \cap \text{Ker } L = \partial\Omega \cap \mathbb{R}^m$, *i.e.*, x is a constant vector in $\mathbb{R}^m$ with $\|x\| = C$. For such x we obtain

$$\|QHx\| \ge \min_{i=\overline{1,m}} \left(\tilde{A}_i - \mathcal{M}_i \sum_{j=1}^{m} \mathcal{B}_{ji}\mathcal{G}_{ji} \right) C - \sum_{i=1}^{m} \rho_i.$$

By condition **H5**

$$\min_{i=\overline{1,m}} \left(\tilde{A}_i - \mathcal{M}_i \sum_{j=1}^{m} \mathcal{B}_{ji}\mathcal{G}_{ji} \right) > 0.$$

Then we can choose $C > C^*$ so large that

$$\min_{i=\overline{1,m}} \left(\tilde{A}_i - \mathcal{M}_i \sum_{j=1}^{m} \mathcal{B}_{ji}\mathcal{G}_{ji} \right) C > \sum_{i=1}^{m} \rho_i.$$

Hence for $x \in \partial\Omega \cap \text{Ker } L$ we have $\|QHx\| > 0$ and $QHx \neq 0$, that is, condition (b) of Lemma 2.1 is satisfied.

To prove (c), we define the mapping $(QH)_\mu :$ Dom $L \times [0,1] \longrightarrow \mathbb{X}$ by $(QH)_\mu = -\mu\tilde{A} + (1-\mu)QH$, where $\tilde{A}x = (\tilde{A}_1x_1, \tilde{A}_2x_2, \ldots, \tilde{A}_mx_m)^T$.

For $x \in \partial\Omega \cap \text{Ker } L$ as above we obtain

$$\|(QH)_\mu x\| \geq \min_{i=\overline{1,m}} \left(\tilde{A}_i - \mathcal{M}_i \sum_{j=1}^{m} \mathcal{B}_{ji}\mathcal{G}_{ji} \right) C - \sum_{i=1}^{m} \rho_i > 0.$$

This means that $(QH)_\mu x \neq 0$ for $x \in \partial\Omega \cap \text{Ker } L$ and $\mu \in [0,1]$. From the homotopy invariance of the Brouwer degree [12] it follows that

$$\deg(QH,\, \Omega \cap \text{Ker } L,\, 0) = \deg(-\tilde{A},\, \Omega \cap \text{Ker } L,\, 0) = (-1)^m \neq 0.$$

According to Lemma 2.1 the equation (6) has at least one N-periodic solution. This completes the proof of Theorem 2.1. □

Theorem 2.2. *Suppose that conditions* ***H1*****–*****H4*****,** ***H6*****,** ***H7*** *hold. Then the N-periodic solution of* (6) *is unique and globally exponentially stable.*

Sketch of the proof. Let $\boldsymbol{g}_{ij}(\ell) = \max\{g_{ij}(\ell),\ c_{ij}(\ell)\}$, $i,j = \overline{1,m}$, $\ell = \overline{1,N}$. Clearly,

$$\sum_{\ell=1}^{N} \boldsymbol{g}_{ij}(\ell) = \mathcal{G}_{ij}, \qquad i,j = \overline{1,m}.$$

Now we shall use the following assertion.

Lemma 2.2. *Assume that condition* ***H6*** *holds. Then there exists $\overline{\lambda} > 1$ such that for any $i = \overline{1,m}$, $n \in I_N$ and $\lambda \in (1, \overline{\lambda}]$ we have*

$$\lambda(1 - A_i(n)) + \mathcal{M}_i \sum_{j=1}^{m} \mathcal{B}_{ji} \sum_{\ell=1}^{N} \lambda^{\ell+1} \boldsymbol{g}_{ji}(\ell) - 1 \leq 0.$$

Now let us suppose that $x^*(n) = (x_1^*(n), x_2^*(n), \ldots, x_m^*(n))^T$ is an N-periodic solution of equation (6), and $x(n) = (x_1(n), x_2(n), \ldots, x_m(n))^T$ is any solution of (6) for $n \geq 0$, defined at least for $n \geq -N$.

From (6) and (7) for $n \in \mathbb{Z}_0^+ = \{n \in \mathbb{Z} :\ n \geq 0\}$, $n \neq n_k$ we derive

$$|x_i(n+1) - x_i^*(n+1)| \leq (1 - A_i(n))|x_i(n) - x_i^*(n)|$$

$$+ \sum_{j=1}^{m} \overline{b}_{ij} M_j \sum_{\ell=1}^{N} g_{ij}(\ell)|x_j(n-\ell) - x_j^*(n-\ell)|,$$

while for $n = n_k$ we have

$$|x_i(n_k+1) - x_i^*(n_k+1)| \leq (1 - A_i(n_k))|x_i(n_k) - x_i^*(n_k)|$$

$$+ \sum_{j=1}^{m} \overline{B}_{ij} L_j \sum_{\ell=1}^{N} c_{ij}(\ell)|x_j(n_k-\ell) - x_j^*(n_k-\ell)|.$$

Now we introduce the quantities

$$y_i(n) = \lambda^n |x_i(n) - x_i^*(n)|, \quad \lambda \in (1, \overline{\lambda}], \quad i = \overline{1,m}, \quad n \geq -N.$$

Then we have

$$y_i(n+1) \leq \lambda(1-A_i(n))y_i(n) + \sum_{j=1}^{m} \mathcal{B}_{ij}\mathcal{M}_j \sum_{\ell=1}^{N} \lambda^{\ell+1} \boldsymbol{g}_{ij}(\ell) y_j(n-\ell), \quad (10)$$

$$\lambda \in (1, \overline{\lambda}], \quad i = \overline{1,m}, \quad n \in \mathbb{Z}_0^+.$$

Now we consider a Lyapunov functional $V(n) = V(y_1, y_2, \dots, y_m)(n)$ defined by

$$V(n) = \sum_{i=1}^{m} \left\{ y_i(n) + \sum_{j=1}^{m} \mathcal{B}_{ij}\mathcal{M}_j \sum_{\ell=1}^{N} \lambda^{\ell+1} \boldsymbol{g}_{ij}(\ell) \sum_{s=n-\ell}^{n-1} y_j(s) \right\}, \quad n \in \mathbb{Z}_0^+.$$

Taking into account (10), we estimate the difference $\Delta V(n) = V(n+1) - V(n)$ for $n \in \mathbb{Z}_0^+$:

$$\Delta V(n) \leq \sum_{i=1}^{m} \left\{ \lambda(1 - A_i(n)) + \mathcal{M}_i \sum_{j=1}^{m} \mathcal{B}_{ji} \sum_{\ell=1}^{N} \lambda^{\ell+1} \boldsymbol{g}_{ji}(\ell) - 1 \right\} y_i(n).$$

By virtue of Lemma 2.2 we have $\Delta V(n) \leq 0$ for all $n \in \mathbb{Z}_0^+$, which implies that

$$V(n) \leq V(0), \qquad n \in \mathbb{Z}_0^+. \quad (11)$$

On the other hand, we have

$$V(n) \geq \sum_{i=1}^{m} y_i(n) = \sum_{i=1}^{m} \lambda^n |x_i(n) - x_i^*(n)|$$

and

$$V(0) \le \sum_{i=1}^{m} \left\{ 1 + \mathcal{M}_i \sum_{j=1}^{m} \mathcal{B}_{ji} \sum_{\ell=1}^{N} \ell \overline{\lambda}^{\ell+1} \boldsymbol{g}_{ji}(\ell) \right\} \max_{s \in I_{-N}} |x_i(s) - x_i^*(s)|,$$

where $I_{-N} = \{-N, -N+1, \ldots, -1, 0\}$. Here we used the fact that $1 < \lambda \le \overline{\lambda}$.

Thus from inequality (11) we obtain

$$\sum_{i=1}^{m} |x_i(n) - x_i^*(n)| \le M\lambda^{-n} \sum_{i=1}^{m} \max_{s \in I_{-N}} |x_i(s) - x_i^*(s)|, \quad n \in \mathbb{Z}_0^+,$$

where

$$M = \max_{i=\overline{1,m}} \left(1 + \mathcal{M}_i \sum_{j=1}^{m} \mathcal{B}_{ji} \sum_{\ell=1}^{N} \ell \overline{\lambda}^{\ell+1} \boldsymbol{g}_{ji}(\ell) \right).$$

This completes the proof of Theorem 2.2. □

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